MAGNETIC FIELD SOLUTIONS FOR ONE CLASS OF NONLINEAR BOUNDARY VALUE PROBLEMS

By

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LIST OF SYMBOLS

Symbol	Description
$\bar{A}$	Vector magnetic potential
A	Magnitude of the z component of $\bar{A}$
$A_{m,n}$	A evaluated at points m and n
$\bar{a}_z$	Unit vector in the z direction
$\bar{B}$	Magnetic flux density vector
B	Magnitude of $\bar{B}$
B_{n1}	Normal component of $\bar{B}$ on side 1 of a boundary
B_o	Gap field
B_x	The x component of $\bar{B}$
δ	Mesh size factor
e	Naperian base 2.71828.....
$\ln N$	Natural log of the number N
$\bar{H}$	Magnetic field intensity vector
H	Magnitude of $\bar{H}$
H_{t1}	Tangential component of $\bar{H}$ on side 1 of a boundary
$\bar{I}$	Volume current density vector
i_z	The z component of $\bar{I}$
i_{ze}	An equivalent i_z

Symbol	Description
Φ	Parameter of flux per unit depth
R	A ratio parameter
μ	Magnetic permeability
μ_o	Permeability of free space
$\mu_{m,n}$	The value of μ at mesh point m,n
ν	The reciprocal of μ
ν'	The derivative of ν with respect to B
$x,y,z,$	Rectangular space coordinates
X	Boundary extremity on x axis
Y	Boundary extremity on y axis

Abstract of Dissertation Presented to the Graduate Council
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Analytical d.c. magnetic field solutions to nonlinear boundary value problems, which result when ferromagnetic materials are placed in magnetic fields, are conspicuously missing from the literature. The work presented in this dissertation presents one approach which yields acceptable results for many of these types of boundary value problems.

Any attempt to solve the general anisotropic nonlinear boundary value problem would most likely be futile. Therefore, a system of analytical techniques was applied to a simplified model, which is a good approximation to many different types of the original more difficult problem. The simplified model essentially consisted of approximating the

anisotropic magnetic medium as a nonhomogeneous magnetic medium. Although the nonhomogeneous boundary value problem is still nonlinear, it is shown that solutions can be obtained.

The B-H loop for annealed low carbon steel was used to obtain the nonhomogeneous magnetic characteristics. The only quantity needed from the B-H loop was the magnetic permeability (μ), which was shown to be representable as a function of magnetic flux density (B).

Maxwell's equations for the time-invariant case were used to derive the two equations which describe the magnetic fields for the nonhomogeneous problem. These equations are put into a more useful form by defining and introducing vector magnetic potential into the equations.

In order to illustrate that the equations are valid, a particular two-dimensional boundary value problem is posed and solved. Three major steps were required to obtain the solutions. First, the equations for vector magnetic potential had to be put into a form which applied to the given two-dimensional problem. The magnetic characteristics had to be put into an analytical form at this point also. Second, boundary conditions which applied to the problem being treated were formulated and presented in a composite form. Third, finite difference schemes were devised for the equations which describe the problem. These schemes were used to convert the problem into a computer program which was solved utilizing the IBM 709 computer.

The magnetic field computer solutions for the particular boundary value problem were interpreted and shown to yield the expected results. It was easy to see that as the sample material was exposed to continuously stronger fields, it became saturated.

Thus, it can be concluded that the major contribution of this dissertation is that new techniques have been generated which can be applied to a whole new class of nonlinear boundary value problems. This class can be described as those d.c. magnetic nonlinear boundary value problems which result when the magnetic medium is either nonhomogeneous or approximately nonhomogeneous.

CHAPTER 1

INTRODUCTION

In recent years microwave technology has been involved in a continual state of growth and advancement due to the demands created by many phases of communication. In the process of meeting these new demands, many related problems are continually confronted.

One such problem, which was brought to the attention of the author by Dr. A. D. Sutherland (of Sperry Microwave Tube Division of the Sperry Rand Corporation of Gainesville, Florida), is that of determining the magnetic field distribution in the electron gun region of traveling wave tubes. High frequency and high power specifications require highly convergent electron beams. This condition results since high frequency dictates smaller physical dimensions of the beam while high power would require higher beam energy densities. If the beam trajectories coincide with the magnetic flux lines, the electron gun problem is reduced to an electrostatic electron gun design. Since the d.c. magnetic focusing circuit is terminated in the electron gun neighborhood, the resulting magnetic field has the required high degree of convergence. Thus, if the magnetic field distribution is known and if the beam trajectories are made to coincide with the magnetic field lines, the electron gun design is greatly simplified. Heretofore, the magnetic field in the gun region has been found from iron filing patterns. However, this method not only requires a previously constructed focusing system, but it also becomes increasingly unreliable for the smaller dimensions that are being encountered. In view of these points, it would be desirable if an analytical method of analysis could be derived.

Thus the problem posed is:

> Given the geometry of a magnetic circuit with a magnetic focusing material, which is not only nonhomogeneous but also anisotropic, to find the magnetic field configuration to a sufficient degree of accuracy by some analytical means.

It should be noted that, although the problem as posed above is an outgrowth of microwave advancements, any solution techniques generated will apply to a much more general class of nonlinear boundary value problems. In general, the d.c. magnetic field problem is a nonlinear and nonhomogeneous boundary value problem with an anisotropic magnetic medium present.

Before a solution is attempted, the anisotropic magnetic properties will be approximated as nonhomogeneous properties. Arguments are presented which validate the approximation. The partial differential equations which apply to each of the media involved (free space, a current region, and a nonhomogeneous magnetic medium) are then derived from Maxwell's equations. Next, a two-dimensional mesh is presented which applies to a particular boundary value problem. Finite difference schemes are devised for the partial differential equations and applied to the chosen boundary value problem. Finally, solutions are obtained by programing the problem on the IBM 709 high-speed digital computer and the results are interpreted to demonstrate that the new solution techniques are valid.

CHAPTER 2

MAGNETIC CHARACTERISTICS

In the most general sense, it would be virtually impossible to depict the complete anisotropic nature of a ferromagnetic material. The behavior of a point in the material is not only dependent on the value of the applied field, but it also depends on the previous and the present states of all the other points in the material. The most fundamental magnetic characteristic is a simple hysteresis loop, which is more commonly referred to as a B-H loop. The B-H loop for a particular sample of annealed low carbon steel is shown in Figure 1 (1) * The characteristics of this sample will be used for the particular boundary value problem that will be treated.

Notice that the state of any differential volume within the sample could be represented by any point on the boundary or inside the B-H loop. A very simple approximation

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*Numbers in parentheses denote entries in the List of References.

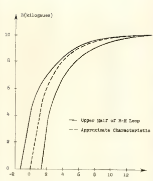

Figure 1. Magnetic Characteristics for Annealed Low Carbon Steel

to the magnetic characteristics would be a single valued function of magnetic flux density vs. magnetic field intensity. Such a characteristic is shown as a dotted line in Figure 1.

There are two reasons why this type of characteristic would be a good approximation. First, the B-H loop is narrow enough so that large errors do not result, and these errors are seen to be much smaller when the sample is driven near saturation. For example, errors of the approximation are less than ten percent for any point in the sample where B is greater than 8 kilogauss. In magnetic focusing structures, materials are usually operated in or near saturation. Second, if such a simplified characteristic were not valid the magnetic field would not repeat when the magnetic circuit was energized at different times and to varying degrees. This behavior has not been observed in practice, or if it has, it has been negligible.

In summary, the anistropic ferromagnetic material characteristics will be approximated as nonhomogeneous magnetic material characteristics. This approach is not only useful for the above reasons, but it will be shown that it is mathematically expedient.

Since the magnetic medium is usually energized and deenergized in one direction only, the single valued characteristic that is used is assumed to lie near the top of the B-H loop. The curve was also chosen from the standpoint of getting a good fit while at the same time having a form easy to treat mathematically. The curve chosen has the form given by

$$B = 4H \qquad 0 \leq H \leq 1 \qquad (2\text{-}1)$$

$$B = 10 - 6e^{-\frac{(H-1)}{1.5}} \qquad 1 \leq H \leq 13.45 \qquad (2\text{-}2)$$

and

$$B = 9.998 + 0.001\,H \qquad H \geq 13.45 \qquad (2\text{-}3)$$

where B is kilogauss and H is in oersteds *

The magnetic permeability, which is defined by

$$\mu = B/H \qquad (2\text{-}4)$$

and which is derived from equations (2-1) through (2-3), is given by

$$\mu = 16\,\pi\,(10^{-4}) \qquad 0 \leq B \leq 0.4 \qquad (2\text{-}5)$$

$$\mu = \frac{40\,\pi B\,(10^{-4})}{1 + 1.5\ln\left(\frac{0.6}{1-B}\right)} \qquad 0.4 \leq B \leq 0.9998 \qquad (2\text{-}6)$$

and

$$\mu = \frac{4\,\pi\,B\,(10^{-7})}{(B - 0.9998) + 13.45(.0001)} \qquad B \geq 0.9998 \qquad (2\text{-}7)$$

---

*Unless otherwise specified, all symbols, which are defined in the List of Symbols, are in the MKS system of units.

The second breakpoint for B was chosen such that the slope of the B-H characteristic was μ_0. This was done since just above saturation, the B-H characteristic becomes a straight line with slope μ_0.

It is shown later that the derivative of ν ($\nu = 1/\mu$) with respect to B (ν') is needed in the problem formulation. Therefore, it is presented at this time.

$$\nu' = 0 \qquad 0 \leq B \leq 0.4 \qquad (2\text{-}8)$$

$$\nu' = \left[\frac{1.5}{B(1-B)} - \frac{1}{B^2} + \frac{1.5 \ln \frac{1-B}{0.6}}{B^2} \right] \cdot \frac{10^3}{4\pi}$$

$$0.4 \leq B \leq 0.9998 \qquad (2\text{-}9)$$

$$\nu' = \frac{7.945\ (10^5)}{B^2} \qquad B \geq 0.9998 \qquad (2\text{-}10)$$

In summary the necessary magnetic characteristics of a particular ferromagnetic material have been formulated and presented in equations (2-6) through (2-10). The functions described by these equations are plotted in Figure 2. The digital computer program that was used to obtain the second breakpoint of B and to solve for μ and ν' as functions of B is given in Appendix A.

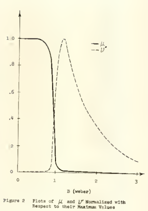

Figure 2 Plots of μ and ν Normalized with Respect to their Maximum Values

CHAPTER 3

MAGNETIC FIELD EQUATIONS

There are certain restrictions for the particular boundary value problem that will be treated. The first of these is that the problem is time invariant; therefore, Maxwell's equations are decoupled and it is only necessary to deal with the two magnetic field relations. Further, the problem will be for an isotropic nonhomogeneous magnetic medium in the presence of free space and current carrying conductors. Of course in free space and in the more common nonferromagnetic conductors,

$$\mu = \mu_0 \quad . \tag{3-1}$$

In the nonhomogeneous magnetic medium, μ and ν will be treated as implicit functions of space coordinates. With these restrictions, the two magnetic field relations of Maxwell's equations become

$$\nabla \cdot \bar{B} = 0 \tag{3-2}$$

$$\nabla \times \bar{H} = \bar{I} \ , \tag{3-3}$$

and the auxiliary relation

$$\bar{B} = \mu \bar{H} \tag{3-4}$$

is understood to be defined for μ a function of space coordinates. Equation (3-4) can be used to change the form of (3-3) to get the set of equations (3-5a) and (3-5b).

$$\nabla \cdot \bar{B} = 0 \tag{3-5a}$$

$$\nabla \times (\nu \bar{B}) = \bar{I} \tag{3-5b}$$

where

$$\nu = \frac{1}{\mu} \tag{3-6}$$

Equations (3-5a) and (3-5b) agree with the magnetic field equations derived by Smythe (2) for the same conditions.

The equations (3-5a) and (3-5b) could be applied in their present form, however, it can be shown that a new vector quantity, the vector magnetic potential $\bar{A}$, is easier to utilize. The defining equations for vector magnetic potential are given by (3,4).

$$\nabla \ \bar{A} = 0 \qquad (3\text{-}7a)$$

$$\nabla \times \bar{A} = \bar{B} \ . \qquad (3\text{-}7b)$$

When equation (3-7b) is substituted into equations (3-5a) and (3-5b), the relations obtained are

$$\nabla \cdot \nabla \times \bar{A} = 0 \qquad (3\text{-}8a)$$

and

$$\nabla \times (\nu \ \nabla \times \bar{A}) = \bar{I} \ . \qquad (3\text{-}8b)$$

If solutions for $\bar{A}$ can be obtained from equations (3-8a) and (3-8b), then equation (3-7b) can be used to find the magnetic field. Note that equation (3-8a) is automatically satisfied by a vector identity (5). Thus, it is only necessary to solve the equation (3-8b) for $\bar{A}$. This simplification in the equations was obtained by using the defining relations (3-7a) and (3-7b).

The solution for the magnetic field boundary value problem that has been outlined has been reduced to that of finding solutions to equation (3-8b), which can be expanded in the form of equation (3-9) or (3-10) by using another vector identity (5).

$$\nu \nabla \times \nabla \times \bar{A} + \nabla \nu \times (\nabla \times \bar{A}) = I \qquad (3\text{-}9)$$

$$\nabla \times \nabla \times \bar{A} = \mu I - \mu \nabla \nu \times (\nabla \times \bar{A}) \qquad (3\text{-}10)$$

If one other vector identity is applied to equation (3-10) (5), then the final relation

$$\nabla^2 \bar{A} = \mu \nabla \nu \times (\nabla \times \bar{A}) - \mu I \qquad (3\text{-}11)$$

is obtained. Equation (3-11) is a general relation and as such is applied to free space, magnetic medium, and current regions.

The equation for vector magnetic potential in free space is given by

$$\nabla^2 \bar{A} = 0 \qquad (3\text{-}12)$$

and it is obtained from equation (3-11) by recognizing that

$$I = 0 \qquad (3\text{-}13)$$

and

$$\mu = \mu_0 \quad \text{or} \quad \nabla \nu = 0 \qquad (3\text{-}14)$$

The equation for the vector magnetic potential in a nonferromagnetic conductor with current I is given by

$$\nabla^2 \bar{A} = -\mu_0 \bar{I} \quad , \tag{3-15}$$

since

$$\nabla \nu = 0 \quad . \tag{3-16}$$

The equation for the vector magnetic potential in the magnetic focusing material is

$$\nabla^2 \bar{A} = \mu \nabla \nu \times (\nabla \times \bar{A}) \tag{3-17}$$

and is obtained from (3-11) since $\bar{I} = 0$.

In summary, three equations

$$\nabla^2 \bar{A} = 0 \tag{3-18}$$

$$\nabla^2 \bar{A} = -\mu \bar{I} \tag{3-19}$$

$$\nabla^2 \bar{A} = \mu \nabla \nu \times (\nabla \times \bar{A}) \tag{3-20}$$

are derived which describe the vector magnetic potential in free space, in a current carrying non-magnetic conductor, and in a nonhomogeneous magnetic medium respectively. These relations will be used, with formulated finite difference schemes, in a computer program to solve a particular nonlinear boundary value problem.

CHAPTER 4

THE SOLUTIONS TO A PARTICULAR BOUNDARY VALUE PROBLEM

The basic relations which are needed to solve one class of nonlinear magnetic field boundary value problems have been derived and presented in Chapters 2 and 3. In the present chapter these techniques will be applied to a particular problem which will illustrate that the derived relations are correct. It should be noted that the application of these relations constitute a major portion of the difficulty in solving problems of the nature outlined.

A problem, which is simple enough to solve without undue expense and time and at the same time complex enough to illustrate nonlinear solutions, is presented in Figure 3. It should be pointed out that this problem was the last of three that were seriously considered. As simple as it is, the problem required one and one-half hours computer time on the IBM 709 computer. A simple description of the problem will serve to clarify Figure 3.

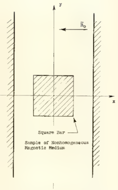

Figure 3. The Cross Section Geometry of the Nonlinear Boundary Value Problem with a Uniform Gap Field $\bar{B}_0$ Applied Between Infinite μ Pole Faces

> What is the two dimensional magnetic field configuration produced when a bar of square cross section with nonhomogeneous magnetic characteristics is inserted in a uniform gap field?

The boundary conditions for the problem depicted in Figure 3 will be given in Section 4-1. The equations that apply to this problem will be presented in Section 4-2. Finally, the results obtained will be presented in Section 4-3.

Section 4-1 -- Boundary Conditions

The boundary conditions could be formulated for the problem as it is presented in Figure 3. However, due to symmetry, it will only be necessary to treat one quadrant of the geometry. This quadrant is presented in Figure 4 along with one typical mesh point.

In order to formulate the boundary conditions, the integral relation (6)

$$\overline{A} = \int_{vol} \frac{\mu \overline{I}}{4 \pi R} \, dv \tag{4-1}$$

should be considered. Note that

$$\overline{I} = \nabla \times \overline{H} \tag{4-2}$$

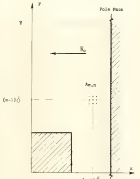

Figure 4 A Cross Section Quadrant of the Particular Boundary Value Problem with a Typical Mesh Point

from one of Maxwell's equations, equation (3-3). Since $\overline{H}$ has only x and y components and is only a function of x and y, $\overline{A}$ has only a z component ($\overline{A} = A\overline{a}_z$) and is a function of x and y. Also, since the x-z plane is a constant flux plane, then it can be chosen as a zero potential reference plane. Further, from the symmetry chosen, the boundary condition for $\overline{A}$ at the y-z plane becomes

$$\frac{\partial A}{\partial x} = 0 \quad . \qquad (4-3)$$

Finally, the boundary condition for the outer extremity (X) of the problem can be obtained from the function

$$\Phi = \int_0^y B_x dy = \int_0^y \frac{\partial A}{\partial y} dy = A(y) \qquad (4-4)$$

since A is only a function of y at x = X. Note that the field is uniform at X; then

$$\Phi = A = B_o y \qquad (4-5)$$

which is the final boundary condition needed. This follows because

$$A = B_0 Y \qquad \text{for } y = Y \tag{4-6}$$

which may be obtained simply from equation (4-5).

For a composite reference, the previously derived boundary conditions are compiled and presented here.

$$A = 0 \qquad y = 0 \qquad 0 \leq x \leq X \tag{4-7a}$$

$$\frac{\partial A}{\partial x} = 0 \qquad x = 0 \qquad 0 \leq y \leq Y \tag{4-7b}$$

$$A = B_0 y \qquad x = X \qquad 0 \leq y \leq Y \tag{4-7c}$$

$$A = B_0 Y \qquad y = Y \qquad 0 \leq x \leq X \tag{4-7d}$$

Section 4-2 -- Equation Formulation

The equations that are used to solve the boundary value problem will first be presented in the partial differential equation form and then in the finite difference form

Partial differential equations can be obtained by a direct application of the problem restrictions to equations (3-18), (3-19), and (3-20) Equation (3-20) will be treated first and then the applicable equations may be compiled by inspection. Since $\overline{A}$ has only a z component and is only a function of x and y, then

$$\nabla^2 \bar{A} = \left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}\right)\bar{a}_z . \qquad (4\text{-}8)$$

Due to the fact that μ and ν for the time being are considered to be implicit functions of space coordinates, (x and y) then

$$\mu \nabla \nu \times (\nabla \times \bar{A}) = -\mu \left[\frac{\partial \nu}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial \nu}{\partial y}\frac{\partial A}{\partial y}\right]\bar{a}_z . \ (4\text{-}9)$$

At this point the magnetic medium characteristics can be taken into account. The magnetic flux density

$$\bar{B} = \nabla \times \bar{A} \qquad (4\text{-}10)$$

has a magnitude given by

$$B = \sqrt{\left(\frac{\partial A}{\partial x}\right)^2 + \left(\frac{\partial A}{\partial y}\right)^2} \quad . \qquad (4\text{-}11)$$

The quantities μ and ν were shown to be representable as functions of B alone by equations (2-5) through (2-10). Thus these relations follow:

$$\frac{\partial \nu}{\partial x} = \nu' \frac{\partial B}{\partial x} \tag{4-12a}$$

$$\frac{\partial \nu}{\partial y} = \nu' \frac{\partial B}{\partial y} \tag{4-12b}$$

The partial derivatives of B with respect to x and y are obtained from equation (4-11)

$$\frac{\partial B}{\partial x} = \frac{1}{B} \left[\frac{\partial A}{\partial x} \frac{\partial^2 A}{\partial x^2} + \frac{\partial A}{\partial y} \frac{\partial^2 A}{\partial x \partial y} \right] \tag{4-13a}$$

and

$$\frac{\partial B}{\partial y} = \frac{1}{B} \left[\frac{\partial A}{\partial x} \frac{\partial^2 A}{\partial x \partial y} + \frac{\partial A}{\partial y} \frac{\partial^2 A}{\partial y^2} \right] \tag{4-13b}$$

These latter relations illustrate that μ and ν satisfy the condition specified earlier, i.e., they are representable as implicit functions of space coordinates.

Equations (4-8) through (4-13) are the relations needed to simplify equation (3-20) to

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = -\frac{\mu \nu'}{B}\left[\frac{\partial A}{\partial x}\left(\frac{\partial A}{\partial x}\frac{\partial^2 A}{\partial x^2}\right.\right.$$

$$\left. + \frac{\partial A}{\partial y}\frac{\partial^2 A}{\partial x \partial y}\right) + \frac{\partial A}{\partial y}\left(\frac{\partial A}{\partial x}\frac{\partial^2 A}{\partial x \partial y}\right.$$

$$\left.\left. + \frac{\partial A}{\partial y}\frac{\partial^2 A}{\partial y^2}\right)\right] \qquad (4\text{-}14)$$

Another form of (4-14) is

$$\frac{\partial^2 A}{\partial x^2}\left[1 + \frac{\mu \nu'}{B}\left(\frac{\partial A}{\partial x}\right)^2\right]$$

$$+ \frac{\partial^2 A}{\partial y^2}\left[1 + \frac{\mu \nu'}{B}\left(\frac{\partial A}{\partial y}\right)^2\right]$$

$$+ \frac{2\mu \nu'}{B}\frac{\partial A}{\partial x}\frac{\partial A}{\partial y}\frac{\partial^2 A}{\partial x \partial y} = 0 \quad . \qquad (4\text{-}15)$$

This relation serves better to illustrate the nonlinear nature of the problem since products and squares are displayed together.

With equations (4-8) and (4-14), the equations which are necessary to describe the particular boundary value problem are given by

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = 0 \tag{4-16a}$$

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = - \frac{\mu \nu'}{B} \left[\frac{\partial A}{\partial x} \left(\frac{\partial A}{\partial x} \frac{\partial^2 A}{\partial x^2} + \frac{\partial A}{\partial y} \frac{\partial^2 A}{\partial x \, \partial y} \right) + \frac{\partial A}{\partial y} \left(\frac{\partial A}{\partial x} \frac{\partial^2 A}{\partial x \, \partial y} + \frac{\partial A}{\partial y} \frac{\partial^2 A}{\partial y^2} \right) \right] \tag{4-16b}$$

where equation (4-16a) refers to the free space region while (4-16b) refers to the magnetic region. If there were currents present in the problem to be treated, then a third relation obtained from equation (3-19) would be

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = - \mu \, i_z \tag{4-17}$$

where the z component of $\overline{B}$ is still zero. Although this relation is not needed, it is worth noting that equations (4-16b) and (4-17) can be used to identify an equivalent z component of current; i.e.,

$$i_{ze} = \frac{\nu'}{B} \left[\frac{\partial A}{\partial x} \left(\frac{\partial A}{\partial x} \frac{\partial^2 A}{\partial x^2} + \frac{\partial A}{\partial y} \frac{\partial^2 A}{\partial x \, \partial y} \right) \right.$$

$$\left. + \frac{\partial A}{\partial y} \left(\frac{\partial A}{\partial x} \frac{\partial^2 A}{\partial x \, \partial y} + \frac{\partial A}{\partial y} \frac{\partial^2 A}{\partial y^2} \right) \right] \qquad (4\text{-}18)$$

This iterates the fact that magnetic polarization effects can be replaced by equivalent currents (6).

The boundaries between the magnetic medium and free space constitute an abrupt change in magnetic permeability. The normal components of magnetic flux density and the tangential components of magnetic field intensity must be matched at these boundaries. The pertinent equations are

$$B_{n_1} = B_{n_2} \qquad (4\text{-}19)$$

$$H_{t_1} = H_{t_2} \quad . \qquad (4\text{-}20)$$

From the definition of $\bar{A}$, that is

$$\nabla \times \bar{A} = \bar{B} \quad , \qquad (4\text{-}21)$$

equations (4-19) and (4-20) can be transformed to

$$\left[\frac{\partial A}{\partial x_t} \right]_1 = \left[\frac{\partial A}{\partial x_t} \right]_2 \qquad (4\text{-}22)$$

and

$$\frac{1}{\mu_1}\left[\frac{\partial A}{\partial x_n}\right]_1 = \frac{1}{\mu_2}\left[\frac{\partial A}{\partial x_n}\right]_2 \quad . \qquad (4\text{-}23)$$

The first of these two relations is automatically satisfied since A is a continuous function throughout the mesh. The second equation is satisfied when the first order derivative approximations are matched on each side of the boundary. The value of A on the boundary can be shown to be a function of adjacent values of A that are not on the boundary itself. The finite difference scheme applied to equation (4-23) will be presented along with other finite difference forms.

All the equations which are necessary to describe the particular boundary value problem have been formulated. These equations (4-16a), (4-16b) and (4-23) can be put into similar finite difference forms by using the first order approximations (7). These are

$$\frac{\partial A}{\partial x} = \frac{A_{m+1,n} - A_{m-1,n}}{2\delta} \qquad (4\text{-}24a)$$

$$\frac{\partial A}{\partial y} = \frac{A_{m,n+1} - A_{m,n-1}}{2\delta} \tag{4-24b}$$

$$\frac{\partial^2 A}{\partial x^2} = \frac{1}{\delta^2}\left[A_{m+1,n} + A_{m-1,n} - 2\,A_{m,n}\right] \tag{4-24c}$$

$$\frac{\partial^2 A}{\partial y^2} = \frac{1}{\delta^2}\left[A_{m,n+1} + A_{m,n-1} - 2\,A_{m,n}\right] \tag{4-24d}$$

$$\frac{\partial^2 A}{\partial x\,\partial y} = \frac{1}{4\,\delta^2}\Big[A_{m+1,n+1} + A_{m-1,n-1}$$

$$- A_{m+1,n-1} - A_{m-1,n+1}\Big] \quad . \tag{4-24e}$$

When equations (4-24) are applied to equations (4-16a) and (4-16b), the results are

$$A_{m,n} = \frac{A_{m+1,n} + A_{m-1,n} + A_{m,n+1} + A_{m,n-1}}{4} \tag{4-25a}$$

$$A_{m,n} = \left[4\left(1 + \frac{\mu \nu' B}{2}\right)\right]^{-1} \Big[A_{m+1,n}$$

$$+ A_{m-1,n} + A_{m,n+1} + A_{m,n-1} + \frac{\mu \nu'}{8\,\delta^2 B} \Big(2(A_{m-1,n}$$

$$+ A_{m-1,n})\,(A_{m+1,n} - A_{m-1,n})^2 + 2\,(A_{m,n+1}$$

$$+ A_{m,n-1}) \quad (A_{m,n+1} - A_{m,n-1})^2$$

$$+ (A_{m+1,n} - A_{m-1,n}) \quad (A_{m,n+1} - A_{m,n-1})^2 (A_{m+1,n+1}$$

$$+ A_{m-1,n-1} - A_{m-1,n+1} - A_{m+1,n-1}) \Big) \Big] \tag{4-25b}$$

The application of equations (4-24) to equation (4-23) for a boundary which is parallel to the x-axis would yield the relation

$$A_{m,n} = \frac{\mu_{m,n+1} A_{m,n-1} + \mu_{m,n-1} A_{m,n+1}}{\mu_{m,n+1} + \mu_{m,n-1}} \tag{4-25c}$$

where other boundary orientations would yield similar relations.

The boundary conditions in equation set (4-7) and finite difference relations in (4-25) constitute the necessary relations that are needed to solve the problem of Figure 4. These relations were converted to a digital computer program which is presented in Appendix B. In essence, the program performs the function of applying the correct finite difference equation to each mesh point in an iterative fashion. The initial values of A, before relaxation, were chosen to correspond to the uniform magnetic field on the boundary at Y.

Section 4-3 -- Results

The boundary value problem illustrated in Figure 3 was solved four different times with the computer program in Appendix B. The four cases are enumerated below.

1. The boundary value problem was first solved by treating the magnetic medium as homogeneous; i.e., $\mu = 4000$ and $\nu' = 0$ for all values of B. This type problem is solved in the literature (7). However, this solution not only serves as a check on the equations, but it should be compared to the computer solutions for other cases. It should be noted that the computer program in Appendix B for this case had to be modified to obtain this particular solution.

2. With the computer program of Appendix B, a solution was obtained with a gap field of 2 kilogauss applied. The result was observed to give a maximum field of 3.94 kilogauss. Thus the nonlinear effects should not be in the solution since $\nu' = 0$ for B less than 4 kilogauss. This would indicate that the results for cases 1 and 2 should be the same. They were identical! Why then, was

the second case treated? Because it illustrates that the nonlinear program would revert to the linear solutions when the iron is not driven into the nonlinear region.

3. The third case was for a gap field of 8 kilogauss. This corresponds to a high degree of nonlinearity.

4. The fourth and final case was solved with a gap field of 10 kilogauss, which is approximately the saturation flux density for the ferromagnetic sample treated.

In order to illustrate these cases, the results obtained for cases 1 and 2 are presented in Figure 5 while the results for cases 3 and 4 are presented in Figures 6 and 7 respectively. Each of these magnetic field plots was obtained when constant contours of the z component of vector magnetic potential were sketched directly on the two dimensional arrays of numbers provided by the computer output. Then these plots were transferred to Figures 5, 6, and 7.

There are primarily two observations that serve to validate the techniques for solving the particular nonlinear problem that was treated. First, as the magnetic field increased, the magnetic flux lines began to distort within the magnetic material. This indicated that μ was becoming nonhomogeneous as the magnetic material began to saturate,

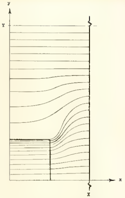

Figure 5. Magnetic Flux Line Solutions for the Nonlinear Boundary Value Problem with a Gap Field of 2 Kilogauss

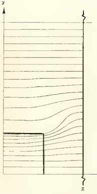

Figure 6. Magnetic Flux Line Solutions for the Nonlinear Boundary Value Problem with a Gap Field of 8 Kilogauss

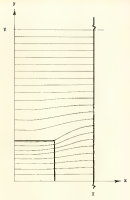

Figure 7- Magnetic Flux Line Solutions for the Nonlinear Boundary Value Problem with a Gap Field of 10 Kilogauss

especially near the corners. The second point, which adds credence to the solutions obtained, is the ratio of the field just inside the magnetic material to that just outside the material. This ratio was obtained at the plane of symmetry where $\bar{B}$ is parallel to the boundary. The resulting ratio parameters have been computed and enumerated below for reference

1. The ratio for cases 1 and 2 is given
   by $R = 10$ (4-26a)
2. The ratio for case 3 is given by
   $R = 3.75$ (4-26b)
   where the gap field was 8 kilogauss.
3. The ratio for case 4 is given by
   $R = 3.00$ (4-26c)
   where the gap field was 10 kilogauss.

These results are exactly as expected; i.e., as the gap field is increased, the magnetic material shows definite trends toward saturation.

CHAPTER 5

CONCLUSIONS

Before any conclusions are presented, it would be appropriate to summarize the work that has been presented. The original problem posed was, "can solutions be obtained for the general anisotropic nonlinear boundary value problem?" The form of the magnetic characteristics allowed the anisotropic magnetic medium to be approximated as a nonhomogeneous medium; i.e., μ was representable as a function of space coordinates. After the vector magnetic potential was introduced, it was applied to Maxwell's d c magnetic equations to obtain a relation which was applicable to a current carrying nonhomogeneous magnetic medium This general relation and the magnetic medium characteristics were used to obtain the equations that would apply to free space, current regions, and nonhomogeneous magnetic regions Then boundary conditions were specified for a particular problem, and this problem was programed and solved on the IBM 709 computer The results obtained, which were positive, were presented in Section 4-3.

The first and most important conclusion to be drawn is that the new techniques applied to this class of nonlinear boundary value problems are valid. This conclusion was established by first treating a linear problem which was simple enough to give known solutions, and then determining the effects of driving the magnetic material into the nonlinear region. The saturation effects observed were exactly those expected.

The second conclusion concerns the complexity of the problem. Although the problem was treated by approximating the anisotropic medium as a nonhomogeneous magnetic medium, this does represent a step in the direction of treating a whole new class of nonlinear boundary value problems. In past work only those problems involving homogeneous magnetic media have been solved. Of course, the usefulness of the techniques will be determined by how well the nonhomogeneous model approximates the anisotropic problem.

There are certain reasons why such a simple problem was treated. First, a problem was chosen for the sole purpose of proving or disproving the techniques applied to this class of nonlinear boundary value problems. Naturally, it was desirable to use a problem which was easy to interpret. Second, as simple as this problem was in form, the applications of the techniques were quite involved. Approximately two hours of computer time were needed to complete the program and solutions.

The techniques that have been presented are probably applicable to the originally posed question, which was to find the fringing magnetic field in the gun region of traveling wave tubes. This conclusion is supported by the fact that the focusing material of a traveling wave tube has a narrow B-H loop as in Figure 1. Also the field plots needed are not in the immediate neighborhood of the iron which is where the errors would be largest.

A final conclusion that can be drawn is that the techniques that are presented here warrant a detailed theoretical and experimental study. Two reasons can be presented which verify this conclusion. First, with all the work and attention that has been focused on the homogeneous problem, it is certainly justifiable to treat in detail the more descriptive nonhomogeneous problem. Second, an extensive study is needed to extend the techniques to axially symmetric problems, three-dimensional problems, and other more general cases.

As a closing remark, it can be concluded that the major contribution of this dissertation is that new techniques have been generated which can be applied to a whole new class of nonlinear boundary value problems. This class can be described as those d.c. magnetic nonlinear boundary value problems which result when the magnetic medium is either nonhomogeneous or approximately nonhomogeneous.

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APPENDIX A

The computer program which is presented here was used to solve for μ and ν^* as a function of B. The defined symbols resemble the quantities they represent closely enough to allow interpretation of the program without a detailed cross-reference.

```
* 145233    FRED O SIMONS JR  *005*NONLINEAR BV PROBLEM
*       XEQ
*       SYMBOL TABLE
C       COMPUTATIONS OF MU,U(ODD), AND NU',U(EVEN).
        DIMENSION U(200)
        HO = 1.+1.5*LOGF(4000.)
        H2UO = HO/(10.**4)
        B2 = 1.-0.6*EXPF((1.-HO)/1.5)
        B=B2
        X= 3./2.
        V=(X /(B*(1.-B))-1./(B*B) - (X *LOGF(0.6/(1.-B)))
       1/B**2)/10.**4
        XK = V*B2**2
        DO 40 M=1,161,2
        XM = M-1
        B = 0.39 +(0.01/2.)*XM
     42 IF(B-0.4) 35,35,37
     35 U(M)   = 4000.
        U(M+1) = 0.
        GO TO 40
     37 IF(B-B2) 38,38,39
     38 U(M)   = (B*10.**4)/(1.+ X*LOGF(0.6/(1.-B)))
        V=(X /(B*(1.-B))-1./(B*B) - (X *LOGF(0.6/(1.-B)))
       1/B**2)/10.**4
        U(M+1) = V
        GO TO 40
     39 U(M)   - B/(B-(B2-H2UO))
        U(M+1)=XK/(B**2)
     40 CONTINUE
        WRITE OUTPUT TAPE 6,1, (U(M),M=1,161)
      1 FORMAT (20X,E12.4,20X,E12.4)
        WRITE OUTPUT TAPE 6,2,HO,H2UO,B2,XK
      2 FORMAT (10X,4E12.4)
        CALL EXIT
        END

                                                        NOFILE
```

APPENDIX B

The computer solution, which was used to solve the particular boundry value problem in Chapter 4, is presented at this time for referance. A complete explanation of the manner in which the problem was programed will not be presented for the sake of brevity, but those interested in the program in detail would not have any difficulty in interpreting the program in its present form.

```
* 145233    FRED O SIMONS JR  *020*NONLINEAR BV PROBLEM
*     XEQ
*     SYMBOL TABLE
C      NON LINEAR BOUNDRY VALUE PROBLEM
      DIMENSION  A(41,81)
      DIMENSION U(21,21)
      S = 5.08/8000.
      FLUX =(1.016/10.**2)
C      FLUX CORRESPONDS TO A FIELD OF 2 KILOGAUSS
C     THIS ROUTINE ASSIGNS INITIAL   VALUES OF A
      DO 21 J=1,81,1
      XJF =J-1
      DO 21 I=1,41,1
   21 A(I,J)= FLUX*(XJF/80.)
      HO = 1.+1.5*LOGF(4000.)
      H2UO = HO/(10.**4)
      B2 = 1.-0.6*EXPF((1.-HO)/1.5)
      B=B2
      X= 3./2.
      V=(X /(B*(1.-B))-1./(B*B) - (X *LOGF(0.6/(1.-B)))
     1/B**2)/10.**4
      XK = V*B2**2
C     LOGIC TO HANDLE NUMBER OF LOOPS
      LCY = 0
   20 LCY = LCY+1
      KCY = 100
      IF(LCY-KCY) 22,22,3
   22 DO 50 N=2,80,1
      DO 50 M=1,40,1
      IF (N-21) 24,25,26
   26 IF(M-1) 60,60,61
   60 A(M,N) = (A(M+1,N)+A(M+1,N)+A(M,N-1)+A(M,N+1))/4.
      GO TO 50
   61 A(M,N) = (A(M+1,N)+A(M-1,N)+A(M,N-1)+A(M,N+1))/4.
      GO TO 50
   25 IF(M-21) 27,28,29
   29 A(M,N) = (A(M+1,N)+A(M-1,N)+A(M,N-1)+A(M,N+1))/4.
      GO TO 50
   28 A(M,N)= (A(20,20)+U(20,20)*A(22,22))/(1.+U(20,20))
      GO TO 50
   27 A(M,N)= (A(M,20)+U(M,20)*A(M,22))/(1.+U(M,20))
      GO TO 50
   24 IF(M-21) 30,31,32
   32 A(M,N) = (A(M+1,N)+A(M-1,N)+A(M,N-1)+A(M,N+1))/4.
      GO TO 50
   31 A(M,N) = (A(20,N)+A(22,N)*U(20,N))/(1.+U(20,N))
      GO TO 50
   30 IF(M-1) 33,33,34
   33 BX = (A(M,N+1) - A(M,N-1))/(2.*S)
      BY = (A(M+1,N) - A(M+1,N))/(2.*S)
      B = SQRTF(BX**2+BY**2)
      IF(B-1./(10.**25)) 41,41,42
   41 A(M,N) = (A(M+1,N)+A(M+1,N)+A(M,N-1)+A(M,N+1))/4.
      U(M,N) = 4000.
   GO TO 50
42 IF(B-0.4) 35,35,37
35 U(M,N) = 4000.
   V = 0.
   GO TO 40
37 IF(B-B2) 38,38,39
38 U(M,N)= (B*10.**4)/(1.+ X*LOGF(0.6/(1.-B)))
   V=(X /(B*(1.-B))-1./(B*B) - (X *LOGF(0.6/(1.-B)))
  1/B**2)/10.**4
   GO TO 40
39 U(M,N) = B/(B-(B2-H2UD))
   V = XK/(B**2)
40 CONTINUE
   XYA = 0.
   EWA = (A(M+1,N) + A(M+1,N))/S**2
   USA = (A(M,N+1)+A(M,N-1))/S**2
   F1 = EWA*BY**2 + USA*BX**2 + 2.*BX*BY*XYA
   F2 = (U(M,N)*V*F1*(S**2))/8
   F3 = A(M+1,N)+A(M+1,N)+A(M,N+1)+A(M,N-1)
   F4 = 4.+U(M,N)*V*B*2.
   A(M,N) = (F3+F2)/F4
   GO TO 50
34 BX = (A(M,N+1) - A(M,N-1))/(2.*S)
   BY = (A(M+1,N) - A(M-1,N))/(2.*S)
   B = SQRTF(BX**2+BY**2)
   IF(B-1./(10.**25)) 51,51,52
51 A(M,N) = (A(M+1,N)+A(M-1,N)+A(N,N-1)+A(M,N+1))/4.
   U(N,N) = 4000.
   GO TO 50
52 IF(B-0.4) 44,44,45
44 U(M,N) = 4000.
   V = 0.
   GO TO 49
45 IF(B-B2) 46,46,47
46 U(M,N)= (B*10.**4)/(1.+ X*LOGF(0.6/(1.-B)))
   V=(X /(B*(1.-B))-1./(B*B) - (X *LOGF(0.6/(1.-B)))
  1/B**2)/10.**4
   GO TO 49
47 U(M,N) = B/(B-(B2-N2UD))
   V = XK/(B**2)
49 CONTINUE
   XYA=(A(M+1,N+1)+A(M-1,N-1)-A(M+1,N-1)-A(M-1,N+1))/
  1(4.*S**2)
   EWA = (A(M+1,N) + A(M-1,N))/S**2
   USA = (A(M,N+1)+A(N,N-1))/S**2
   F1 = EWA*BY**2 + USA*BX**2 + 2.*BX*BY*XYA
   F2 = (U(M,N)*V*F1*(S**2))/8
   F3 = A(M+1,M)+A(N-1,M)+A(M,N+1)+A(N,N-1)
   F4 = 4.+U(N,N)*V*B*2.
   A(M,N) = (F3+F2)/F4
50 CONTINUE
   GO TO 20
 3 WRITE OUTPUT TAPE 6,1, ((A(N,N),M=1,10),N=1,81)
   WRITE OUTPUT TAPE 6,1, ((A(N,N),N=11,20),N=1,81)
```

```
      WRITE OUTPUT TAPE 6,1, ((A(M,N),M=21,30),N=1,81)
      WRITE OUTPUT TAPE 6,2, ((A(M,N),M=31,41),N=1,81)
    1 FORMAT(10X,10E10.3)
    2 FORMAT(10X,11E10.3)
      CALL EXIT
      END
                                                            NOFILE
```

BIOGRAPHICAL SKETCH

Fred Oliver Simons, Jr. was born July 7, 1937, at Heidelberg, Mississippi. He was graduated from Pascagoula High School of Pascagoula, Mississippi, in 1955. After graduation he worked with Mississippi Power Company until August, 1956. In September, 1956, he entered Mississippi State University. He was graduated in August of 1960 from Mississippi State University with the degree Bachelor of Science in Electrical Engineering. He was an instructor at Mississippi State University from September, 1960, until June, 1961, and he worked with the National Aeronautics and Space Administration from May, 1961, until August, 1961. In August, 1961, he entered the Graduate School of the University of Florida where he received his Master of Science in Engineering in June, 1962. Since then he has been working toward the Doctor of Philosophy degree in the Department of Electrical Engineering.

Fred Oliver Simons, Jr. has been married since 1957 to the former Miss Kathleen C. Ivy and is the father of one child. He is a member of Tau Beta Pi, Phi Kappa Phi, Eta Kappa Nu, and the Institute of Electrical and Electronics Engineers.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Engineering and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

June 22, 1965

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